Regularity in C^*-algebras and topological dynamics

Wilhelm Winter

WWU Münster

ECNU Shanghai, 14.5.2013

C*-algebraic regularity

Some classification results

Dynamic regularity: $\mathbb{Z}$-actions

Towards other group actions

DEFINITION (W–Zacharias)

Let A be a C*-algebra, $n \in \mathbb{N}$. We say A has nuclear dimension at most n, $\dim_{\mathrm{nuc}} A \leq n$, if the following holds:

DEFINITION (W–Zacharias)

Let A be a C*-algebra, $n \in \mathbb{N}$. We say A has nuclear dimension at most n, $\dim_{\mathrm{nuc}} A \leq n$, if the following holds:

For any $\mathcal{F} \subset A$ finite and any $\varepsilon > 0$ there is an approximation

$$A \xrightarrow{\psi} F \xrightarrow{\varphi} A$$

DEFINITION (W–Zacharias)
Let A be a C*-algebra, $n \in \mathbb{N}$. We say A has nuclear dimension at most n, $\dim_{\text{nuc}} A \leq n$, if the following holds:

For any $\mathcal{F} \subset A$ finite and any $\varepsilon > 0$ there is an approximation

$$A \xrightarrow{\psi} F \xrightarrow{\varphi} A$$

with F finite dimensional, ψ c.p.c., φ c.p. and

$$\varphi \circ \psi =_{\mathcal{F},\varepsilon} \text{id}_A,$$

DEFINITION (W–Zacharias)
Let A be a C*-algebra, $n \in \mathbb{N}$. We say A has nuclear dimension at most n, $\dim_{\mathrm{nuc}} A \leq n$, if the following holds:

For any $\mathcal{F} \subset A$ finite and any $\varepsilon > 0$ there is an approximation

$$A \xrightarrow{\psi} F \xrightarrow{\varphi} A$$

with F finite dimensional, ψ c.p.c., φ c.p. and

$$\varphi \circ \psi =_{\mathcal{F},\varepsilon} \mathrm{id}_A,$$

and such that F can be written as

$$F = F^{(0)} \oplus \ldots \oplus F^{(n)}$$

with c.p.c. order zero maps

$$\varphi^{(i)} := \varphi|_{F^{(i)}}.$$

DEFINITION/PROPOSITION (using Toms–W, Rørdam–W)
A unital C*-algebra A is $\mathcal{Z}$-stable if and only if for every $K \in \mathbb{N}$ there are c.p.c. order zero maps

$$\Phi : M_K \to A_\infty \cap A'$$

and

$$\Psi : M_2 \to A_\infty \cap A'$$

DEFINITION/PROPOSITION (using Toms–W, Rørdam–W)
A unital C*-algebra A is $\mathcal{Z}$-stable if and only if for every $K \in \mathbb{N}$ there are c.p.c. order zero maps

$$\Phi : M_K \to A_\infty \cap A'$$

and

$$\Psi : M_2 \to A_\infty \cap A'$$

such that

$$\Psi(e_{11}) = \mathbf{1} - \Phi(\mathbf{1}_{M_K})$$

and

$$\Phi(e_{11})\Psi(e_{22}) = \Psi(e_{22})\Phi(e_{11}) = \Psi(e_{22}).$$

DEFINITION

A unital simple C*-algebra A has m-comparison, if whenever $0 \neq a, b \in M_\infty(A)_+$ satisfy

$$d_\tau(a) < d_\tau(b)$$

for all $\tau \in QT(A)$, then

$$a \precsim 1_{m+1} \otimes b$$

DEFINITION

A unital simple C*-algebra A has m-comparison, if whenever $0 \neq a, b \in M_\infty(A)_+$ satisfy

$$d_\tau(a) < d_\tau(b)$$

for all $\tau \in QT(A)$, then

$$a \precsim 1_{m+1} \otimes b$$

(i.e., there is $(s_n)_\mathbb{N} \subset M_\infty(A)$ with $s_n^*(1_{m+1} \otimes b)s_n \stackrel{n\to\infty}{\longrightarrow} a$).

DEFINITION

A unital simple C*-algebra A has m-comparison, if whenever $0 \neq a, b \in M_\infty(A)_+$ satisfy

$$d_\tau(a) < d_\tau(b)$$

for all $\tau \in QT(A)$, then

$$a \precsim 1_{m+1} \otimes b$$

(i.e., there is $(s_n)_{\mathbb{N}} \subset M_\infty(A)$ with $s_n^*(1_{m+1} \otimes b)s_n \stackrel{n\to\infty}{\longrightarrow} a$).

A has comparison, if it has 0-comparison.

C*-algebraic regularity

Some classification results

Dynamic regularity: $\mathbb{Z}$-actions

Towards other group actions

Finite nuclear dimension, $\mathcal{Z}$-stability and comparison are often equivalent; they are particularly useful for classification:

Finite nuclear dimension, $\mathcal{Z}$-stability and comparison are often equivalent; they are particularly useful for classification:

THEOREM (Toms–W, 2009; using results by Lin, W, Lin–Niu, Lin–Phillips, Strung–W,...)
Let

$$\begin{aligned}\mathcal{E} \quad = \quad & \{\mathcal{C}(X) \rtimes_{\beta} \mathbb{Z} \mid X \text{ compact, metrizable, infinite,} \\ & \beta \text{ induced by a minimal homeomorphism,} \\ & \mathrm{K}_0 \text{ separates traces}\}.\end{aligned}$$

Finite nuclear dimension, $\mathcal{Z}$-stability and comparison are often equivalent; they are particularly useful for classification:

THEOREM (Toms–W, 2009; using results by Lin, W, Lin–Niu, Lin–Phillips, Strung–W,...)
Let

$$\begin{aligned}\mathcal{E} \quad = \quad & \{\mathcal{C}(X) \rtimes_\beta \mathbb{Z} \mid X \text{ compact, metrizable, infinite,} \\ & \beta \text{ induced by a minimal homeomorphism,} \\ & \mathrm{K}_0 \text{ separates traces}\}.\end{aligned}$$

For any $A \in \mathcal{E}$, $\dim_{\mathrm{nuc}} A < \infty \Longleftrightarrow A$ is $\mathcal{Z}$-stable $\Longleftrightarrow A$ has comparison.

Finite nuclear dimension, $\mathcal{Z}$-stability and comparison are often equivalent; they are particularly useful for classification:

THEOREM (Toms–W, 2009; using results by Lin, W, Lin–Niu, Lin–Phillips, Strung–W,...)
Let

$$\begin{aligned} \mathcal{E} \quad = \quad & \{\mathcal{C}(X) \rtimes_\beta \mathbb{Z} \mid X \text{ compact, metrizable, infinite,} \\ & \beta \text{ induced by a minimal homeomorphism,} \\ & \mathrm{K}_0 \text{ separates traces}\}. \end{aligned}$$

For any $A \in \mathcal{E}$, $\dim_{\mathrm{nuc}} A < \infty \Longleftrightarrow A$ is $\mathcal{Z}$-stable $\Longleftrightarrow A$ has comparison.

Moreover, the regularity properties ensure classification by ordered K-theory in this case.

Strategy of proof:

Strategy of proof:

Show finite nuclear dimension.

Strategy of proof:

Show finite nuclear dimension.

Derive $\mathcal{Z}$-stability.

Strategy of proof:

Show finite nuclear dimension.
Derive $\mathcal{Z}$-stability.
Derive that classification up to UHF-stability is enough.

Strategy of proof:

Show finite nuclear dimension.
Derive $\mathcal{Z}$-stability.
Derive that classification up to UHF-stability is enough.
Apply Berg's technique to show that classification of $A_x = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{x\}))$ up to UHF-stability is enough.

Strategy of proof:

Show finite nuclear dimension.
Derive $\mathcal{Z}$-stability.
Derive that classification up to UHF-stability is enough.
Apply Berg's technique to show that classification of
$A_x = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{x\}))$ up to UHF-stability is enough.
Observe that $A_x \otimes$ UHF has finite decomposition rank and real rank 0.

Strategy of proof:

Show finite nuclear dimension.
Derive $\mathcal{Z}$-stability.
Derive that classification up to UHF-stability is enough.
Apply Berg's technique to show that classification of
$A_x = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{x\}))$ up to UHF-stability is enough.
Observe that $A_x \otimes$ UHF has finite decomposition rank and real rank 0.
Derive that $A_x \otimes$ UHF is TAF and apply Lin's TAF classification.

Strategy of proof:

Show finite nuclear dimension.
Derive $\mathcal{Z}$-stability.
Derive that classification up to UHF-stability is enough.
Apply Berg's technique to show that classification of
$A_x = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{x\}))$ up to UHF-stability is enough.
Observe that $A_x \otimes$ UHF has finite decomposition rank and real rank 0.
Derive that $A_x \otimes$ UHF is TAF and apply Lin's TAF classification.

QUESTIONS

(i) What if projections do not separate traces?

Strategy of proof:

Show finite nuclear dimension.
Derive $\mathcal{Z}$-stability.
Derive that classification up to UHF-stability is enough.
Apply Berg's technique to show that classification of
$A_x = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{x\}))$ up to UHF-stability is enough.
Observe that $A_x \otimes$ UHF has finite decomposition rank and real rank 0.
Derive that $A_x \otimes$ UHF is TAF and apply Lin's TAF classification.

QUESTIONS

(i) What if projections do not separate traces?
(E.g. for not uniquely ergodic minimal homeomorphisms of odd spheres.)

Strategy of proof:

Show finite nuclear dimension.
Derive $\mathcal{Z}$-stability.
Derive that classification up to UHF-stability is enough.
Apply Berg's technique to show that classification of
$A_x = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{x\}))$ up to UHF-stability is enough.
Observe that $A_x \otimes$ UHF has finite decomposition rank and real rank 0.
Derive that $A_x \otimes$ UHF is TAF and apply Lin's TAF classification.

QUESTIONS

(i) What if projections do not separate traces?
(E.g. for not uniquely ergodic minimal homeomorphisms of odd spheres.)

(ii) What about $\mathbb{Z}^d$-actions?

Strategy of proof:

Show finite nuclear dimension.
Derive $\mathcal{Z}$-stability.
Derive that classification up to UHF-stability is enough.
Apply Berg's technique to show that classification of
$A_x = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{x\}))$ up to UHF-stability is enough.
Observe that $A_x \otimes$ UHF has finite decomposition rank and real rank 0.
Derive that $A_x \otimes$ UHF is TAF and apply Lin's TAF classification.

QUESTIONS

(i) What if projections do not separate traces?
(E.g. for not uniquely ergodic minimal homeomorphisms of odd spheres.)

(ii) What about $\mathbb{Z}^d$-actions?
(E.g. for free minimal $\mathbb{Z}^d$-actions on Cantor sets.)

Strategy of proof:

Show finite nuclear dimension.
Derive $\mathcal{Z}$-stability.
Derive that classification up to UHF-stability is enough.
Apply Berg's technique to show that classification of $A_x = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{x\}))$ up to UHF-stability is enough.
Observe that $A_x \otimes$ UHF has finite decomposition rank and real rank 0.
Derive that $A_x \otimes$ UHF is TAF and apply Lin's TAF classification.

QUESTIONS

(i) What if projections do not separate traces?
(E.g. for not uniquely ergodic minimal homeomorphisms of odd spheres.)

(ii) What about $\mathbb{Z}^d$-actions?
(E.g. for free minimal $\mathbb{Z}^d$-actions on Cantor sets.)

For (i), replace TAF by TASomething (e.g. TAI). This works in principle, but showing that $A_x \otimes$ UHF is TASomething can be hard.

Strategy of proof:

Show finite nuclear dimension.
Derive $\mathcal{Z}$-stability.
Derive that classification up to UHF-stability is enough.
Apply Berg's technique to show that classification of
$A_x = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{x\}))$ up to UHF-stability is enough.
Observe that $A_x \otimes$ UHF has finite decomposition rank and real rank 0.
Derive that $A_x \otimes$ UHF is TAF and apply Lin's TAF classification.

QUESTIONS

(i) What if projections do not separate traces?
(E.g. for not uniquely ergodic minimal homeomorphisms of odd spheres.)

(ii) What about $\mathbb{Z}^d$-actions?
(E.g. for free minimal $\mathbb{Z}^d$-actions on Cantor sets.)

For (i), replace TAF by TASomething (e.g. TAI). This works in principle, but showing that $A_x \otimes$ UHF is TASomething can be hard.
For (ii), the problem is more severe, as it is not clear how to replace A_x.

The next result in particular implies that, in the presence of finite nuclear dimension, it is enough to find tracially uniformly large approximate embeddings into classifiable C*-algebras:

The next result in particular implies that, in the presence of finite nuclear dimension, it is enough to find tracially uniformly large approximate embeddings into classifiable C^*-algebras:

THEOREM (W, 2013)
Let S be F or I (finite dimensional C^*-algebras or interval algebras).

The next result in particular implies that, in the presence of finite nuclear dimension, it is enough to find tracially uniformly large approximate embeddings into classifiable C*-algebras:

THEOREM (W, 2013)
Let S be F or I (finite dimensional C*-algebras or interval algebras). Let A be a separable, simple, unital C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$ and $T(A) \neq \emptyset$, and let

$$\left(A \xrightarrow{\sigma_i} B_i \xrightarrow{\varrho_i} A\right)_{i \in \mathbb{N}}$$

be a system of maps with the following properties:

The next result in particular implies that, in the presence of finite nuclear dimension, it is enough to find tracially uniformly large approximate embeddings into classifiable C*-algebras:

THEOREM (W, 2013)
Let S be F or I (finite dimensional C*-algebras or interval algebras). Let A be a separable, simple, unital C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$ and $T(A) \neq \emptyset$, and let

$$\left(A \xrightarrow{\sigma_i} B_i \xrightarrow{\varrho_i} A\right)_{i \in \mathbb{N}}$$

be a system of maps with the following properties:

1. $B_i \in$ S, $i \in \mathbb{N}$

The next result in particular implies that, in the presence of finite nuclear dimension, it is enough to find tracially uniformly large approximate embeddings into classifiable C*-algebras:

THEOREM (W, 2013)
Let S be F or I (finite dimensional C*-algebras or interval algebras).
Let A be a separable, simple, unital C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$ and $T(A) \neq \emptyset$, and let

$$\left(A \xrightarrow{\sigma_i} B_i \xrightarrow{\varrho_i} A\right)_{i \in \mathbb{N}}$$

be a system of maps with the following properties:

1. $B_i \in$ S, $i \in \mathbb{N}$
2. ϱ_i is an embedding for each $i \in \mathbb{N}$

The next result in particular implies that, in the presence of finite nuclear dimension, it is enough to find tracially uniformly large approximate embeddings into classifiable C*-algebras:

THEOREM (W, 2013)
Let S be F or I (finite dimensional C*-algebras or interval algebras).
Let A be a separable, simple, unital C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$ and $T(A) \neq \emptyset$, and let

$$\big(A \xrightarrow{\sigma_i} B_i \xrightarrow{\varrho_i} A\big)_{i \in \mathbb{N}}$$

be a system of maps with the following properties:

1. $B_i \in$ S, $i \in \mathbb{N}$
2. ϱ_i is an embedding for each $i \in \mathbb{N}$
3. σ_i is c.p.c. for each $i \in \mathbb{N}$

The next result in particular implies that, in the presence of finite nuclear dimension, it is enough to find tracially uniformly large approximate embeddings into classifiable C*-algebras:

THEOREM (W, 2013)
Let S be F or I (finite dimensional C*-algebras or interval algebras). Let A be a separable, simple, unital C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$ and $T(A) \neq \emptyset$, and let

$$\left(A \xrightarrow{\sigma_i} B_i \xrightarrow{\varrho_i} A\right)_{i \in \mathbb{N}}$$

be a system of maps with the following properties:

1. $B_i \in$ S, $i \in \mathbb{N}$
2. ϱ_i is an embedding for each $i \in \mathbb{N}$
3. σ_i is c.p.c. for each $i \in \mathbb{N}$
4. $\bar{\sigma} : A \to \prod_{\mathbb{N}} B_i / \bigoplus_{\mathbb{N}} B_i$ induced by the σ_i is a unital *-homomorphism

The next result in particular implies that, in the presence of finite nuclear dimension, it is enough to find tracially uniformly large approximate embeddings into classifiable C*-algebras:

THEOREM (W, 2013)
Let S be F or I (finite dimensional C*-algebras or interval algebras). Let A be a separable, simple, unital C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$ and $T(A) \neq \emptyset$, and let

$$\left(A \xrightarrow{\sigma_i} B_i \xrightarrow{\varrho_i} A\right)_{i \in \mathbb{N}}$$

be a system of maps with the following properties:

1. $B_i \in$ S, $i \in \mathbb{N}$
2. ϱ_i is an embedding for each $i \in \mathbb{N}$
3. σ_i is c.p.c. for each $i \in \mathbb{N}$
4. $\bar{\sigma} : A \to \prod_{\mathbb{N}} B_i / \bigoplus_{\mathbb{N}} B_i$ induced by the σ_i is a unital *-homomorphism
5. $\sup\{|\tau(\varrho_i\sigma_i(a) - a)| \mid \tau \in T(A)\} \xrightarrow{i \to \infty} 0$ for each $a \in A$.

The next result in particular implies that, in the presence of finite nuclear dimension, it is enough to find tracially uniformly large approximate embeddings into classifiable C*-algebras:

THEOREM (W, 2013)
Let S be F or I (finite dimensional C*-algebras or interval algebras). Let A be a separable, simple, unital C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$ and $T(A) \neq \emptyset$, and let

$$\left(A \xrightarrow{\sigma_i} B_i \xrightarrow{\varrho_i} A\right)_{i \in \mathbb{N}}$$

be a system of maps with the following properties:

1. $B_i \in$ S, $i \in \mathbb{N}$
2. ϱ_i is an embedding for each $i \in \mathbb{N}$
3. σ_i is c.p.c. for each $i \in \mathbb{N}$
4. $\bar{\sigma} : A \to \prod_{\mathbb{N}} B_i / \bigoplus_{\mathbb{N}} B_i$ induced by the σ_i is a unital *-homomorphism
5. $\sup\{|\tau(\varrho_i \sigma_i(a) - a)| \mid \tau \in T(A)\} \stackrel{i \to \infty}{\longrightarrow} 0$ for each $a \in A$.

Then, $A \otimes$ UHF is TAS.

Strategy of proof:

Hypotheses yield an embedding

$$\lambda : A \otimes \mathrm{UHF} \to (A \otimes \mathrm{UHF})_\infty$$

which is TAS (as a map).

Strategy of proof:

Hypotheses yield an embedding

$$\lambda : A \otimes \mathrm{UHF} \to (A \otimes \mathrm{UHF})_\infty$$

which is TAS (as a map).
The image

$$\lambda(A \otimes \mathrm{UHF}) \subset (A \otimes \mathrm{UHF})_\infty$$

is in general position, but one can use finite nuclear dimension and strict comparison to move it into a position compatible with the canonical embedding

$$A \otimes \mathrm{UHF} \subset (A \otimes \mathrm{UHF})_\infty.$$

Strategy of proof:

Hypotheses yield an embedding

$$\lambda : A \otimes \mathrm{UHF} \to (A \otimes \mathrm{UHF})_\infty$$

which is TAS (as a map).
The image

$$\lambda(A \otimes \mathrm{UHF}) \subset (A \otimes \mathrm{UHF})_\infty$$

is in general position, but one can use finite nuclear dimension and strict comparison to move it into a position compatible with the canonical embedding

$$A \otimes \mathrm{UHF} \subset (A \otimes \mathrm{UHF})_\infty .$$

Now the canonical embedding will be TAS as well.

COROLLARY (Special case of Matui–Sato, 2013)
Let A be a separable, simple, unital, monotracial C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$. Suppose that A is quasidiagonal.
Then, $A \otimes$ UHF is TAF.

COROLLARY (Special case of Matui–Sato, 2013)
Let A be a separable, simple, unital, monotracial C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$. Suppose that A is quasidiagonal.
Then, $A \otimes \mathrm{UHF}$ is TAF.

PROOF
Quasidiagonality yields approximate maps

$$A \otimes \mathrm{UHF} \to \mathcal{Q}.$$

COROLLARY (Special case of Matui–Sato, 2013)
Let A be a separable, simple, unital, monotracial C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$. Suppose that A is quasidiagonal.
Then, $A \otimes$ UHF is TAF.

PROOF
Quasidiagonality yields approximate maps

$$A \otimes \mathrm{UHF} \to \mathcal{Q}.$$

Choose tracially large embeddings

$$\mathcal{Q} \to \mathrm{UHF} \to A \otimes \mathrm{UHF}$$

and apply the theorem. ∎

Our technique also applies in the case of larger trace spaces:

Our technique also applies in the case of larger trace spaces:

THEOREM

Let A be a separable, simple, unital C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$ and real rank zero; suppose $\partial_e T(A)$ is nonempty and compact.

Our technique also applies in the case of larger trace spaces:

THEOREM

Let A be a separable, simple, unital C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$ and real rank zero; suppose $\partial_e T(A)$ is nonempty and compact.
Suppose further that for each $\tau \in \partial_e T(A)$ there are a simple, unital, monotracial AF algebra D with trace δ and a unital embedding

$$\beta : A \to D$$

with

$$\delta \circ \beta = \tau.$$

Our technique also applies in the case of larger trace spaces:

THEOREM
Let A be a separable, simple, unital C*-algebra with $\dim_{\mathrm{nuc}} A < \infty$ and real rank zero; suppose $\partial_e T(A)$ is nonempty and compact.
Suppose further that for each $\tau \in \partial_e T(A)$ there are a simple, unital, monotracial AF algebra D with trace δ and a unital embedding

$$\beta : A \to D$$

with

$$\delta \circ \beta = \tau.$$

Then, $A \otimes \mathcal{Q}$ is TAF.

COROLLARY

Let X be a metrizable compact Hausdorff space with finite covering dimension and let $\alpha : \mathbb{Z}^d \to \text{Homeo}(X)$ be a free, minimal action with compact space of ergodic measures.

COROLLARY

Let X be a metrizable compact Hausdorff space with finite covering dimension and let $\alpha : \mathbb{Z}^d \to \mathrm{Homeo}(X)$ be a free, minimal action with compact space of ergodic measures.
Suppose further that projections in $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^d$ separate traces.

COROLLARY

Let X be a metrizable compact Hausdorff space with finite covering dimension and let $\alpha : \mathbb{Z}^d \to \mathrm{Homeo}(X)$ be a free, minimal action with compact space of ergodic measures.
Suppose further that projections in $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^d$ separate traces.

Then, $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^d$ is rationally TAF. In particular, crossed products of this form are classified by their ordered K-theory.

COROLLARY

Let X be a metrizable compact Hausdorff space with finite covering dimension and let $\alpha : \mathbb{Z}^d \to \mathrm{Homeo}(X)$ be a free, minimal action with compact space of ergodic measures.
Suppose further that projections in $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^d$ separate traces.

Then, $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^d$ is rationally TAF. In particular, crossed products of this form are classified by their ordered K-theory.

Trace preserving AF-embeddings were provided by Lin.

COROLLARY

Let X be a metrizable compact Hausdorff space with finite covering dimension and let $\alpha : \mathbb{Z}^d \to \mathrm{Homeo}(X)$ be a free, minimal action with compact space of ergodic measures.
Suppose further that projections in $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^d$ separate traces.

Then, $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^d$ is rationally TAF. In particular, crossed products of this form are classified by their ordered K-theory.

Trace preserving AF-embeddings were provided by Lin.

To get finite nuclear dimension we need dynamic versions of our regularity properties.

C* -algebraic regularity

Some classification results

Dynamic regularity: $\mathbb{Z}$-actions

Towards other group actions

DEFINITION (based on Hirshberg–W–Zacharias)
Let X be compact, metrizable, infinite, and $\alpha : \mathbb{Z} \curvearrowright X$ an action.

DEFINITION (based on Hirshberg–W–Zacharias)
Let X be compact, metrizable, infinite, and $\alpha : \mathbb{Z} \curvearrowright X$ an action. We say $(X, \mathbb{Z}, \alpha)$ has dynamic dimension at most n, $\dim(X, \mathbb{Z}, \alpha) \leq n$, if the following holds:

DEFINITION (based on Hirshberg–W–Zacharias)
Let X be compact, metrizable, infinite, and $\alpha : \mathbb{Z} \curvearrowright X$ an action. We say $(X, \mathbb{Z}, \alpha)$ has dynamic dimension at most n, $\dim(X, \mathbb{Z}, \alpha) \leq n$, if the following holds:
For any open cover $\mathcal{U}$ of X and any $L \in \mathbb{N}$, there is a system

$$(U_{k,l}^{(i)} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{0, \ldots, L\})$$

of open subsets

DEFINITION (based on Hirshberg–W–Zacharias)
Let X be compact, metrizable, infinite, and $\alpha : \mathbb{Z} \curvearrowright X$ an action. We say $(X, \mathbb{Z}, \alpha)$ has dynamic dimension at most n, $\dim(X, \mathbb{Z}, \alpha) \leq n$, if the following holds:
For any open cover $\mathcal{U}$ of X and any $L \in \mathbb{N}$, there is a system

$$(U_{k,l}^{(i)} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{0, \ldots, L\})$$

of open subsets such that

- $\alpha_1(U_{k,l}^{(i)}) = U_{k,l+1}^{(i)}$ for $i \in \{0, \ldots, n\}$, $k \in \{1, \ldots, K^{(i)}\}$, $l \in \{0, \ldots, L-1\}$

DEFINITION (based on Hirshberg–W–Zacharias)
Let X be compact, metrizable, infinite, and $\alpha : \mathbb{Z} \curvearrowright X$ an action. We say $(X, \mathbb{Z}, \alpha)$ has dynamic dimension at most n, $\dim(X, \mathbb{Z}, \alpha) \leq n$, if the following holds:
For any open cover $\mathcal{U}$ of X and any $L \in \mathbb{N}$, there is a system

$$(U_{k,l}^{(i)} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{0, \ldots, L\})$$

of open subsets such that

- $\alpha_1(U_{k,l}^{(i)}) = U_{k,l+1}^{(i)}$ for $i \in \{0, \ldots, n\}$, $k \in \{1, \ldots, K^{(i)}\}$, $l \in \{0, \ldots, L-1\}$
- for each fixed $i \in \{0, \ldots, n\}$ the sets $U_{k,l}^{(i)}$ are pairwise disjoint

DEFINITION (based on Hirshberg–W–Zacharias)
Let X be compact, metrizable, infinite, and $\alpha : \mathbb{Z} \curvearrowright X$ an action. We say $(X, \mathbb{Z}, \alpha)$ has dynamic dimension at most n, $\dim(X, \mathbb{Z}, \alpha) \leq n$, if the following holds:
For any open cover $\mathcal{U}$ of X and any $L \in \mathbb{N}$, there is a system

$$(U_{k,l}^{(i)} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{0, \ldots, L\})$$

of open subsets such that

- $\alpha_1(U_{k,l}^{(i)}) = U_{k,l+1}^{(i)}$ for $i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{0, \ldots, L-1\}$
- for each fixed $i \in \{0, \ldots, n\}$ the sets $U_{k,l}^{(i)}$ are pairwise disjoint
- $(U_{k,l}^{(i)} \mid i \in \{0, \ldots, n\}, k \in \{1, \ldots, K^{(i)}\}, l \in \{0, \ldots, L\})$ is an open cover of X refining $\mathcal{U}$.

REMARKS

- We think of $n+1$ as the number of colors, of $K^{(i)}$ as the number of towers of color i, and of L as the length of the towers.

REMARKS

- We think of $n+1$ as the number of colors, of $K^{(i)}$ as the number of towers of color i, and of L as the length of the towers.
- In a similar vein, one can define dynamic versions of comparison and of $\mathcal{Z}$-stability.

REMARKS

- We think of $n+1$ as the number of colors, of $K^{(i)}$ as the number of towers of color i, and of L as the length of the towers.
- In a similar vein, one can define dynamic versions of comparison and of $\mathcal{Z}$-stability.
- The three notions are closely related, especially in the minimal, uniquely ergodic case.

REMARKS

- We think of $n+1$ as the number of colors, of $K^{(i)}$ as the number of towers of color i, and of L as the length of the towers.
- In a similar vein, one can define dynamic versions of comparison and of $\mathcal{Z}$-stability.
- The three notions are closely related, especially in the minimal, uniquely ergodic case.
- For $\mathbb{Z}^d$-actions, one simply replaces $\{0,\ldots,L\}$ by $\{0,\ldots,L\}^d$.

THEOREM (Szabó, 2013; generalizing Hirshberg–W–Zacharias, 2011)

Let $(X, \mathbb{Z}^d, \alpha)$ be free.

THEOREM (Szabó, 2013; generalizing Hirshberg–W–Zacharias, 2011)

Let $(X, \mathbb{Z}^d, \alpha)$ be free.
If $\dim X < \infty$, then $\dim(X, \mathbb{Z}^d, \alpha) < \infty$.

THEOREM (Szabó, 2013; generalizing Hirshberg–W–Zacharias, 2011)

Let $(X, \mathbb{Z}^d, \alpha)$ be free.
If $\dim X < \infty$, then $\dim(X, \mathbb{Z}^d, \alpha) < \infty$.

As a consequence, $\dim_{\mathrm{nuc}}(\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^d)$ is finite.

THEOREM (Szabó, 2013; generalizing Hirshberg–W–Zacharias, 2011)

Let $(X, \mathbb{Z}^d, \alpha)$ be free.
If dim $X < \infty$, then dim$(X, \mathbb{Z}^d, \alpha) < \infty$.

As a consequence, $\dim_{\text{nuc}}(\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^d)$ is finite.

The proof uses Gutman's marker property and Lindenstrauss' topological small boundary property. The arguments for $\mathbb{Z}$ and for $\mathbb{Z}^d$ are not that much different.

C*-algebraic regularity

Some classification results

Dynamic regularity: $\mathbb{Z}$-actions

Towards other group actions

What about more general groups?

What about more general groups?

For G finitely generated with word length metric, one might use $B_L(e)$ in place of $\{0, \ldots, L\}$.

What about more general groups?

For G finitely generated with word length metric, one might use $B_L(e)$ in place of $\{0, \ldots, L\}$. In this case, there is a nice *relative* result:

What about more general groups?

For G finitely generated with word length metric, one might use $B_L(e)$ in place of $\{0, \ldots, L\}$. In this case, there is a nice *relative* result:

THEOREM (Bartels–Lück–Reich)
Let G be a hyperbolic group acting on its Rips complex $\bar{X}$
(G acts freely, $\bar{X}/G$ is compact, $\bar{X}$ is contractible).

What about more general groups?

For G finitely generated with word length metric, one might use $B_L(e)$ in place of $\{0,\ldots,L\}$. In this case, there is a nice *relative* result:

THEOREM (Bartels–Lück–Reich)
Let G be a hyperbolic group acting on its Rips complex $\bar{X}$ (G acts freely, $\bar{X}/G$ is compact, $\bar{X}$ is contractible).

Then, there is $d \in \mathbb{N}$ such that the following holds:
For any $L \in \mathbb{N}$ there is an open cover $\mathcal{U}$ of $G \times \bar{X}$ satisfying

What about more general groups?

For G finitely generated with word length metric, one might use $B_L(e)$ in place of $\{0, \ldots, L\}$. In this case, there is a nice *relative* result:

THEOREM (Bartels–Lück–Reich)
Let G be a hyperbolic group acting on its Rips complex $\bar{X}$ (G acts freely, $\bar{X}/G$ is compact, $\bar{X}$ is contractible).

Then, there is $d \in \mathbb{N}$ such that the following holds:
For any $L \in \mathbb{N}$ there is an open cover $\mathcal{U}$ of $G \times \bar{X}$ satisfying

- $\mathcal{U}$ has covering number (or dimension) at most d

What about more general groups?

For G finitely generated with word length metric, one might use $B_L(e)$ in place of $\{0, \ldots, L\}$. In this case, there is a nice *relative* result:

THEOREM (Bartels–Lück–Reich)
Let G be a hyperbolic group acting on its Rips complex $\bar{X}$ (G acts freely, $\bar{X}/G$ is compact, $\bar{X}$ is contractible).

Then, there is $d \in \mathbb{N}$ such that the following holds:
For any $L \in \mathbb{N}$ there is an open cover $\mathcal{U}$ of $G \times \bar{X}$ satisfying

- $\mathcal{U}$ has covering number (or dimension) at most d
- for every $x \in \bar{X}$, $B_L(e) \times \{x\} \subset U$ for some $U \in \mathcal{U}$

What about more general groups?

For G finitely generated with word length metric, one might use $B_L(e)$ in place of $\{0, \ldots, L\}$. In this case, there is a nice *relative* result:

THEOREM (Bartels–Lück–Reich)
Let G be a hyperbolic group acting on its Rips complex $\bar{X}$ (G acts freely, $\bar{X}/G$ is compact, $\bar{X}$ is contractible).

Then, there is $d \in \mathbb{N}$ such that the following holds:
For any $L \in \mathbb{N}$ there is an open cover $\mathcal{U}$ of $G \times \bar{X}$ satisfying

- $\mathcal{U}$ has covering number (or dimension) at most d
- for every $x \in \bar{X}$, $B_L(e) \times \{x\} \subset U$ for some $U \in \mathcal{U}$
- for every $g \in G$ and $U \in \mathcal{U}$, $gU \in \mathcal{U}$

What about more general groups?

For G finitely generated with word length metric, one might use $B_L(e)$ in place of $\{0, \ldots, L\}$. In this case, there is a nice *relative* result:

THEOREM (Bartels–Lück–Reich)
Let G be a hyperbolic group acting on its Rips complex $\bar{X}$ (G acts freely, $\bar{X}/G$ is compact, $\bar{X}$ is contractible).

Then, there is $d \in \mathbb{N}$ such that the following holds:
For any $L \in \mathbb{N}$ there is an open cover $\mathcal{U}$ of $G \times \bar{X}$ satisfying

- $\mathcal{U}$ has covering number (or dimension) at most d
- for every $x \in \bar{X}$, $B_L(e) \times \{x\} \subset U$ for some $U \in \mathcal{U}$
- for every $g \in G$ and $U \in \mathcal{U}$, $gU \in \mathcal{U}$
- for every $g \in G$ and $U \in \mathcal{U}$, either $gU = U$ or $gU \cap U = \emptyset$

What about more general groups?

For G finitely generated with word length metric, one might use $B_L(e)$ in place of $\{0, \ldots, L\}$. In this case, there is a nice *relative* result:

THEOREM (Bartels–Lück–Reich)
Let G be a hyperbolic group acting on its Rips complex $\bar{X}$ (G acts freely, $\bar{X}/G$ is compact, $\bar{X}$ is contractible).

Then, there is $d \in \mathbb{N}$ such that the following holds:
For any $L \in \mathbb{N}$ there is an open cover $\mathcal{U}$ of $G \times \bar{X}$ satisfying

- $\mathcal{U}$ has covering number (or dimension) at most d
- for every $x \in \bar{X}$, $B_L(e) \times \{x\} \subset U$ for some $U \in \mathcal{U}$
- for every $g \in G$ and $U \in \mathcal{U}$, $gU \in \mathcal{U}$
- for every $g \in G$ and $U \in \mathcal{U}$, either $gU = U$ or $gU \cap U = \emptyset$
- for every $U \in \mathcal{U}$, the subgroup $G_U = \{g \in G \mid gU = U\}$ is virtually cyclic (contains a cyclic subgroup with finite index).

What about more general groups?

For G finitely generated with word length metric, one might use $B_L(e)$ in place of $\{0, \ldots, L\}$. In this case, there is a nice *relative* result:

THEOREM (Bartels–Lück–Reich)
Let G be a hyperbolic group acting on its Rips complex $\bar{X}$ (G acts freely, $\bar{X}/G$ is compact, $\bar{X}$ is contractible).

Then, there is $d \in \mathbb{N}$ such that the following holds:
For any $L \in \mathbb{N}$ there is an open cover $\mathcal{U}$ of $G \times \bar{X}$ satisfying

- $\mathcal{U}$ has covering number (or dimension) at most d
- for every $x \in \bar{X}$, $B_L(e) \times \{x\} \subset U$ for some $U \in \mathcal{U}$
- for every $g \in G$ and $U \in \mathcal{U}$, $gU \in \mathcal{U}$
- for every $g \in G$ and $U \in \mathcal{U}$, either $gU = U$ or $gU \cap U = \emptyset$
- for every $U \in \mathcal{U}$, the subgroup $G_U = \{g \in G \mid gU = U\}$ is virtually cyclic (contains a cyclic subgroup with finite index).

(This plays a crucial role in their proof of the Farrell–Jones conjecture for hyperbolic groups.)

For $\mathbb{Z}$-actions, we can characterize finite dynamic dimension along these lines:

For $\mathbb{Z}$-actions, we can characterize finite dynamic dimension along these lines:

THEOREM (Bartels–W)
For $(X, \mathbb{Z}, \alpha)$, $\dim(X, \mathbb{Z}, \alpha) < \infty$ if and only if there is $d \in \mathbb{N}$ such that the following holds:

For $\mathbb{Z}$-actions, we can characterize finite dynamic dimension along these lines:

THEOREM (Bartels–W)
For $(X, \mathbb{Z}, \alpha)$, $\dim(X, \mathbb{Z}, \alpha) < \infty$ if and only if there is $d \in \mathbb{N}$ such that the following holds:

For any $L \in \mathbb{N}$ there is an open cover $\mathcal{U}$ of $G \times X$ satisfying

For $\mathbb{Z}$-actions, we can characterize finite dynamic dimension along these lines:

THEOREM (Bartels–W)
For $(X, \mathbb{Z}, \alpha)$, $\dim(X, \mathbb{Z}, \alpha) < \infty$ if and only if there is $d \in \mathbb{N}$ such that the following holds:

For any $L \in \mathbb{N}$ there is an open cover $\mathcal{U}$ of $G \times X$ satisfying

- $\mathcal{U}$ has covering number (or dimension) at most d
- for every $x \in X$, $B_L(e) \times \{x\} \subset U$ for some $U \in \mathcal{U}$
- for every $g \in G$ and $U \in \mathcal{U}$, $gU \in \mathcal{U}$
- for every $0 \neq g \in G$ and $U \in \mathcal{U}$, $gU \cap U = \emptyset$, i.e., for every $U \in \mathcal{U}$, the subgroup $G_U = \{g \in G \mid gU = U\}$ is trivial.